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On the existence of a combinatorial Schlegel diagram of a simplicial unstacked 3-polytope with a prescribed set of vertices

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Abstract

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It is shown that, except for two well defined configurations, any finite set $V \subset \mathbb{R}^2$ with exactly three points on ∂V is the vertex set of a triangulation of $[V]$ which is combinatorially the Schlegel diagram of a simplicial unstacked 3-polytope.

0. Introduction

A *Schlegel diagram* of a 3-polytope P is a central projection of its boundary cell-complex on one of its facets F through a point (the center of the projection) which lies beyond F, but closely enough to F. Thus it may be viewed as a cell decomposition of F. If P is simplicial, then F is a triangle and a Schlegel diagram of P on F is a triangulation of F. If P is simplicial and unstacked, then this triangulation T, is *unstacked*, i.e., every subcomplex of T whose union is homomorphic to D^2 ($=$2-disk) is not isometric (rigidly equivalent) to a triangle, except when either the subcomplex is itself a 2-cell of T, or it is the whole of T. Denote by $V(T)$ the vertex set of T; if P is simplicial, then the convex hull of $V(T)$, $[V(T)]=F$ has only 3 points of $V(T)$ on its boundary (the vertices of F). But not every triangulation T (whose 1-cells are straight line segments, and) whose union is a (euclidean) triangle s.t. the 0-skeleton ($=$vertex set) of T lies entirely inside the triangle, except for the three boundary vertices, is

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a Schlegel diagram of a simplicial 3-polytope (in the concrete sense defined above; see 'Final remark' at the end of the paper). Hence by *unstacked* triangulation T of a triangle we mean here not necessarily that it is a concrete Schlegel diagram of an unstacked simplicial 3-polytope, but in the somewhat weaker i.e., combinatorial sense that no subcomplex whose union is a topological 2-disc ($=D^2$) is isometric to a (euclidean) triangle, except when either the subcomplex is itself a 2-cell of T, or it is the whole of T. By Steiner's Theorem every such a triangulation is combinatorially isomorphic to some Schlegel diagram of a simplicial unstacked 3-polytop, hence the word combinatorial in the title. (We always assume that the 1-cells are straight line segments.) The question posed here is the following: given a finite set V of points in the plane such that the convex hull $[V]$ of V contains precisely three boundary points of V (the vertices of $[V]$), is there an unstacked triangulation T of $[V]$ whose set of vertices is V (the 'prescribed set' in the title)? The answer depends on the structure of V. But it turns out that the sets V for which the answer is negative can be classified neatly as follows: either V minus one of its boundary points is in a convex position (in the weak sense), or V minus one of its inner points is contained in the three segments joining this inner point to the three vertices of $[V]$. This will be proved below (Theorem 3.1).

1. The two exceptional cases

We assume throughout that V is a noncollinear planar set of cardinality $5 \leqslant \# V < \infty$, whose convex hull $[V]$ contains only 3 boundary points of V, i.e., $\#(V \cap \partial[V]) = 3$. Let x_1, x_2, x_3 be the vertices of $[V]$. V is *nearly convex* if $V_i := V \setminus \{x_i\}$ is in a convex position (weak sense, i.e., $V_i = V_i \cap \partial[V_i]$) for some $1 \leqslant i \leqslant 3$. V is *completely stacked* if there is a point $u \in V \setminus \{x_1, x_2, x_3\}$ s.t. V is contained on the union of the three segments $[u, x_i]$. Note that for $\# V = 5, 6$ being completely stacked implies (being) nearly convex.

Claim 1.1. *If V is either nearly convex or completely stacked, then there is no unstacked triangulation of $[V]$ whose vertex set is V.*

Remark 1.2. A triangulation whose union is a convex set is called a *convex triangulation*. The 1-dimensional skeleton (edges) of a convex triangulation T with a prescribed set of vertices, say W (the union of T is $[W]$) may be characterized as follows: it is maximal (under inclusion) among sets of (closed) straight line segments with endpoints in W which do not intersect each other, except for a common endpoint (which belongs to W). This 'obvious' result has no obvious 3-space analogue (e.g., to characterize in a similar fashion the 2-dimensional skeleton of a 3-dimensional simplicial complex with a prescribed set of vertices W, and whose union is $[W]$), and it appears to be mentioned in one of the papers of Lawson which is referred to in [1], without proof (the exact proof can be quite tedious).

It follows that any set of segments with endpoints in W which do not intersect each other, except for a common endpoint, is a part of the 1-dimensional skeleton of a convex triangulation whose vertex set is W. Thus, given W, if one finds a segment e with endpoints in W s.t. no other segment with endpoints in W intersects e in its relative interior, then surely any convex triangulation whose 0-dimensional skeleton is W must have e as a 1-cell. This remark can be useful.

Definition 1.3. If W is finite and $u, v \in W$, then the segment $[u, v]$ is *cellular* (relatively to W) provided it does not contain any point of W in its relative interior.

Proof of Claim 1.1. Let T be a (convex) triangulation of $[V]$ whose 0-skeleton is V. Assume first that V is nearly convex, say $V_1 = V \setminus \{x_1\}$ is in a convex position (in the weak sense). Clearly T must have all segments $[x_1, v]: v \in V_1$ as 1-cells (by Remark 1.2). Similarly, every boundary cellular segment of V_1 is a 1-cell in T. Since $\# V_1 \geqslant 4$ ($\# V \geqslant 5$ by assumption) and V_1 is in convex position, there is a 1-cell of T say e which is a strict diagonal of $[V_1]$. Clearly the triangle $[e, x_1]$ contains at least one point of V in its interior, hence T is stacked.

Assume now that V is completely stacked; say $u \in V \setminus \{x_1, x_2, x_3\}$ and $V \subset \bigcup_{i=1}^{3} [u, x_i]$. Denote by x_i' $(1 \leqslant i \leqslant 3)$ the point of $V \cap\,]u, x_i]$ (half open segment) which is nearest to u. It is readily seen that the segments $[x_1', x_2']$, $[x_2', x_3']$, $[x_3', x_1']$ are 1-cells in any triangulation on V (by Remark 1.2), and since $u \in \operatorname{int}[x_1', x_2', x_3']$ any triangulation on V is stacked. □

2. Unstacked convex triangulations with at least 4 boundary 0-cells

Until now we spoke about unstacked triangulation of a set V assuming that $\#(V \cap (\partial[V]) = 3$. In order to deal with the question posed in the beginning we have to widen a bit the setting by deleting the assumption that $\#(V \cap \partial[V]) = 3$. A planar convex triangulation T whose 0-dimensional skeleton V has at least 4 points on $\partial[V]$ is *unstacked* provided every subcomplex whose union is a (convex euclidean) triangle (which is in particular, a topological 2-disc), is a 2-cell (triangle) of the complex. (In case $\#(V \cap \partial[V]) = 3$ this topological 2-disc could also be the whole of $[V]$, which is a (convex euclidean) triangle, without forbiding T of being unstacked).

Theorem 2.1. *Any finite non-collinear planar set V s.t. $\#(V \cap \partial[V]) \geqslant 4$ is the vertex set of an unstacked convex triangulation.*

This will follow from Theorem 2.2 below, which says the same but in a more specific way that will be used later. Here is an additional notation: If a, b, c are three non-collinear points we denote by $ab \to c$ the *open* half plane bounded by the line $\overleftrightarrow{ab}$, which contains c.

Theorem 2.2. *Let V be a non-collinear planar set s.t. $\#(V \cap \partial[V]) \geqslant 4$.*

(a) *If V is not in a weak convex position (i.e., if $V \cap \operatorname{int}[V] \neq \emptyset$) then there exists an unstacked convex triangulation T on V s.t. no diagonal of $[V]$ is a 1-cell of T.*

(b) *If V is in a weak convex position (i.e., if $V = V \cap \partial[V]$) and if v_1, v_2, v_3 are three vertices of $[V]$, then there exists an (unstacked) convex triangulation T on V s.t. for every permutation (i, j, k) of $(1, 2, 3)$ every inner 1-cell of T incident with a point (of V) in $\mathbb{R}^2 \setminus (v_i v_j \rightarrow v_k)$ has its other vertex in $v_i v_j \rightarrow v_k$.*

Clearly Theorem 2.1 is just a shortened version of Theorem 2.2; before proving Theorem 2.2 we need the following.

Definition 2.3. Let W be a finite planar set, and let $[a, b]$ be an edge of $[W]$, i.e., a, b are incident vertices of $[W]$ (possibly some points of W lie in the relative interior of $[a, b]$). If $W \not\subset [a, b]$ define

(2.1) $\Gamma(W) := \partial[W] \setminus [a, b]$, and $W' := W \setminus \Gamma(W)$.

If $W \subset [a, b]$ define $\Gamma(W) :=]a, b[$ (an open segment) and $W' = \{a, b\}$ $(= W \setminus \Gamma(W))$. Note that in either case $a, b \in W'$, and $[a, b]$ is an edge of $[W']$, hence $\Gamma(W')$ is defined. $\Gamma(W)$ is an open polygonal path whose vertex set is $W \setminus W'$ (a, b are terminal points of $\Gamma(W)$ but they do not belong to it).

Lemma 2.4. *Assume W is finite, planar, $[a, b]$ is an edge of $[W]$, $[W] \not\subset [a, b]$ and let $\delta(W)$ be the closed polygonal path formed by $\Gamma(W) \cup \Gamma(W') \cup \{a, b\}$. Assume that $\#(W') \geqslant 3$ (this is automatically satisfied if $W \not\subset [a, b]$, i.e., if W is not in a weak convex position). Then there is a triangulation T of $\delta(W) \cup \operatorname{int} \delta(W)$ (the interior of $\delta(W)$ is taken in the sense of Jordan's Theorem for polygones), whose 0-dimensional skeleton is $W \cap \delta(W)$, such that any inner 1-cell of T (i.e., not lying on $\delta(W)$) has one vertex in $\Gamma(W)$ and one vertex is $\Gamma(W')$ (in particular, no inner 1-cell is incident with a nor with b).*

Remark. $\delta(W)$ is a simple closed polygonal path with the shape of a crescent, and no point of W lies inside $\delta(W)$ (cf. Fig. 1).

Proof. Let T be a triangulation of $\delta(W)$ whose vertex set is $W \cap \delta(W)$ (any simple closed polygon can be triangulated; see, e.g., [2, p. 286, Theorem B-2-1]), and assume that among all the triangulations of $\delta(W)$ T has minimal number of 'bad' 1-cells, i.e., inner 1-cells both of whose endpoints are in $\Gamma(W) \cup \{a, b\}$ (clearly no segment both of whose endpoints are in $\Gamma(W') \cup \{a, b\}$ can be an inner 1-cell of T). We have to show that this minimal number is no more than 0. Assume to the contrary that T has bad 1-cells, and among these choose the longest one, say $e = [u, v]$ ($u, v \in \Gamma(W) \cup \{a, b\}$).

Since e is inner it is a common edge of two 2-cells, say $[u, v, p]$ and $[u, v, q]$. The line spanned by e, aff(e), divides the plane into two open half planes H^+, H^-, one of which, say H^-, entirely contains $\Gamma(W')$ and the other one H^+ contains a connected part of $\Gamma(W)$, $H^+ \cap \Gamma(W)$. Clearly p, q lie on different sides of aff(e), say $p \in H^+ \cap \Gamma(W)$ and $q \in H^-$ (see Fig. 2).

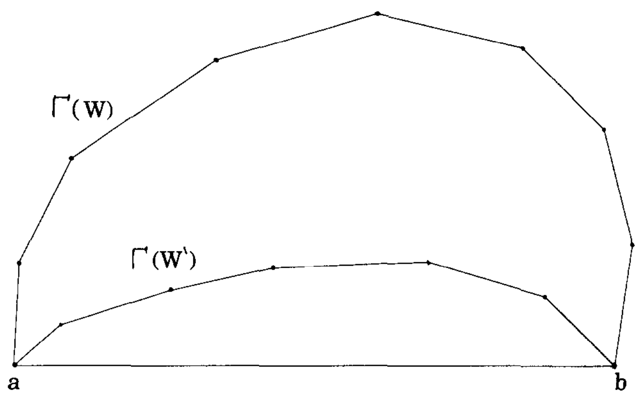

Fig. 1.

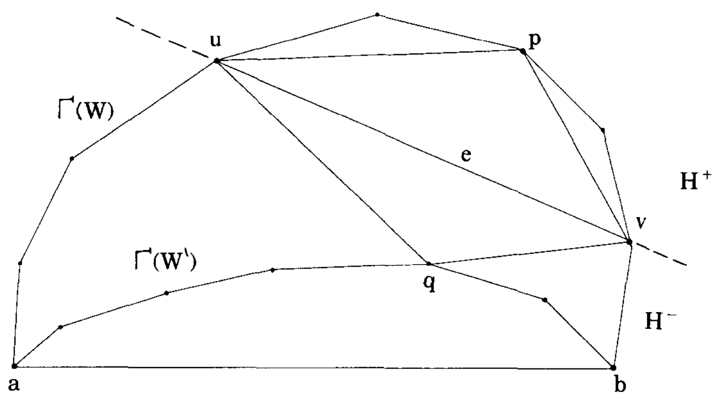

Fig. 2.

Claim 1. $q \in \Gamma(W')$.

Proof. Otherwise, if $q \in H^- \cap (\Gamma(W) \cup \{a, b\})$, then at least one of the 1-cells $[u, q]$, $[v, q]$ is bad and longer than e, contradiction. (Note that here we invoke the assumption that $\# W' \geqslant 3$, otherwise one of the segments $[u, q]$, $[v, q]$ could be $[a, b]$ which is not inner, hence cannot be bad (although it can be longer than e).) □

Claim 2. *$[u, v, p] \cup [u, v, q]$ is a convex quadrangle.*

Proof. *$\Gamma(W')$ lies entirely in the interior of the angle $\angle upv$, and $q \in \Gamma(W')$* (Claim 1). □

Replace now the bad 1-cell $e=[u, v]$ of T by the segment $[p, q]$ which is cellular by Claim 2, to obtain a new triangulation of $\delta(W)$, denoted by T. By Claim 1 $[p, q]$ is not a bad cell, hence T' has less bad 1-cells than T, contradiction. □

Lemma 2.5. *Let V be a finite planar set in convex position (weak sense), and let $[a, b]$ be a diagonal of $[V]$ (i.e., a, b are non-incident vertices of $[V]$, and hence $[V]$ is not a triangle). Then there exists a convex triangulation T of $[V]$ whose 0-skeleton is V s.t. every non-boundary 1-cell of T has its two vertices in different open sides of (the line spanned by $[a, b]$)* aff(a, b). *(In particular, each inner 1-cell of T is incident neither with a nor with b.)*

Proof. The proof is easy and left to the reader; just one hint: one can use the 'flipping of diagonal' argument used in the previous proof to obtain a triangulation with less 'bad' 1-cells. □

Proof of Theorem 2.2(a). By induction on $\# V$. Since $\#(V \cap \partial[V]) \geqslant 4$ and V is not in a weak convex position we have $\# V \geqslant 5$, hence the initial step is for $\# V=5$. In this case there is a unique point $v \in V$ which lies in the interior of $[V]$; join v by 4 segments to the other points of V to obtain the desired unstacked triangulation.

Induction step: Since $\#(V \cap \partial[V]) \geqslant 4$ there are two points $a, b \in V \cap \partial[V]$ s.t. (the relative interior of $[a, b]$) relint$[a, b]$ is contained in (the interior of $[V]$) int$[V]$. Denote by $H^+ H^-$ the *closed* half-planes determined by aff(a, b), and put $V^+=H^+ \cap V$, $V^-=H^- \cap V$ $(a, b \in V^+ \cap V^-)$.

If V^+ is not in a weak convex position, then put $W=V^+$ and $V'=V \backslash \Gamma(W)$. By Lemma 2.4 we can triangulate $\delta(W)$ by an unstacked triangulation $T(\delta(W))$ whose 0-dimensional skeleton is $W \cap \delta(W)$, and every inner 1-cell has one vertex in $\Gamma(W)$ and one vertex in $\Gamma(W')$. Let $T(V')$ be a convex unstacked triangulation of $[V']$ whose 0-dimensional skeleton is V' s.t. either.

(i) if V' is not in convex position, $T(V')$ does not have any diagonal of $[V']$ as a 1-cell; this triangulation is assured by the induction hypothesis, or

(ii) if V' is in a convex position every inner 1-cell of $T(V')$ has its two vertices on different open sides of aff(a, b); this triangulation is assured by Lemma 2.5.

It is readily seen that in either case $T := T(\delta(W)) \cup T(V')$ is an unstacked triangulation of $[V]$ satisfying the requirements of the theorem.

So we may assume that V^+ is in a weak convex position; similarly we may assume that V^- is in such a position. Since V is not in convex position it follows that $(\text{relint}[a, b]) \cap V \neq \emptyset$, hence V^+ satisfies the conditions for W in Lemma 2.4 (including $\#((V^+)') \geqslant 3$). Let T^+ be a triangulation of $\delta(V^+)$ of the kind assured by Lemma 2.4, and similarly let T^- be a triangulation of $\delta(V^-)$ of the kind assured by the same lemma. It is readily seen that $T := T^+ \cup T^-$ is a triangulation of $[V]$ of the desired kind.

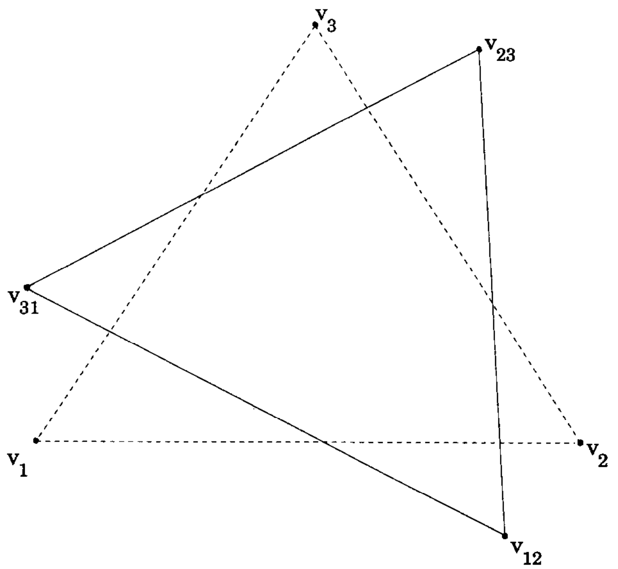

Fig. 3.

Proof of 2.2(b). For every permutation (i, j, k) of $(1,2,3)$ put $V_{ij} = V \cap (\mathbb{R}^2 \setminus (v_i v_j \to v_k))$ (note that $v_i, v_j \in V_{ij}$).

Case 1: *There are* $1 \leqslant i, j \leqslant 3$, $i \neq j$ *s.t.* $V_{ij} = \{v_i, v_j\}$.

Assume, w.l.o.g., that $V_{12} = \{v_1, v_2\}$, i.e., v_1 and v_2 are incident vertices of $[V]$ and $]v_1, v_2[\cap V = \emptyset$. Since $\# V \geqslant 4$ at least one of the sets V_{23}, V_{31} has cardinality $\geqslant 3$, say $\# V_{31} \geqslant 3$; let q be the point of $V_{31} \cap \partial[V]$ which differs from v_3 and is nearest to v_3 (v_3 and q are incident for the cyclic order on $\partial[V]$, and $q \neq v_1$). Construction of T: the inner 1-cells of T are all the segments of the form $[q, v]$ $v \in V_{23}$, and $[v_2, v]$ $v \in V_{31}$, excluding $[v_2, v_3]$ if $V_{23} \setminus \{v_2, v_3\} \neq \emptyset$. It is easy to check that T satisfies the desired properties (to the reader).

Case 2: *For all* $1 \leqslant i, j \leqslant 3$, $i \neq j$ $\# V_{ij} \geqslant 3$.

Let $v_{ij} \in V_{ij} \setminus \{v_i, v_j\}$ be the neighbour of v_j which precede it counterclockwise on $\partial[V]$ (for $1 \leqslant i, j \leqslant 3$, $i \neq j$; see Fig. 3). By assumption $v_{ij} \neq v_i, v_j$. Construction of T: the inner 1-cells of T are all the segments of the form $[v_{12}, v]$ for $v \in V_{23} \setminus \{v_3\}$, $[v_{23}, v]$ for $v \in V_{31} \setminus \{v_1\}$, and $[v_{31}, v]$ for $v \in V_{12} \setminus \{v_2\}$ (Fig. 3; to the reader). $\square$

3. Main result

Theorem 3.1. *Let V be a planar set which is not nearly convex, and not completely stacked. There is an unstacked convex triangulation T of V whose 0-skeleton is V.*

Proof. By Theorem 2.1 we may assume, that $\#(V \cap \partial[V])=3$.

Let v_1, v_2, v_3 be the vertices of $[V]$, and put $V_i = V \setminus \{v_i\}$ for $i=1, 2, 3$. Assume first that for some $1 \leqslant i \leqslant 3$ $\#(V_i \cap \partial[V_i]) \geqslant 4$, say, w.l.o.g., $\#(V_1 \cap \partial[V_1]) \geqslant 4$. Since V is not nearly convex, V_1 is not in a weak convex position. By Theorem 2.2(a) there is an unstacked convex triangulation T_1 of $[V_1]$, whose 0-skeleton is V_1, which does not have any inner 1-cell both of whose vertices are on $\partial[V_1]$. Add now all the segments of the form $[v_1, v]$ $v \in V_1 \cap \partial[V_1]$ to obtain a triangulation T of $[V]$ with the desired properties (to the reader).

Assume henceforth that $\#(V_i \cap \partial[V_i])=3$ for $i=1, 2, 3$. For $1 \leqslant i \leqslant 3$ let v_i' be the point of V_i nearest to v_i; in other words $\{v_i'\} = (V_i \cap \partial[V_i]) \setminus \{v_{i+1}, v_{i+2}\}$ (all indices are taken modulu 3). Clearly the relative interiors of the segments $[v_i', v_{i+1}]$ and $[v_i', v_{i+2}]$ do not contain any point of V $(1 \leqslant i \leqslant 3)$. Similarly the nonconvex quadrangle $(v_i, v_{i+1}, v_i', v_{i+2})$ $(1 \leqslant i \leqslant 3)$ does not contain any point of V, except for its four vertices (Fig. 4). Denote by w_i $(1 \leqslant i \leqslant 3)$ the intersection of the segments $[v_{i+1}, v_{i+2}'][v_{i+2}, v_{i+1}']$ (Fig. 4). Clearly $w_1, w_2, w_3 \notin V$ but $V' := V \setminus \{v_1, v_2, v_3\}$ is contained in the (convex) hexagon $[v_1', w_3, v_2', w_1, v_3', w_2]$. The structure of T will depend heavily on that of V'; split this into 5 cases.

Case 1: *V' is in a weak convex position, and $\# V' \geqslant 4$.*
Case 2: *V' is not in a weak convex position, and $\#(V' \cap \partial[V']) \geqslant 4$.*
Case 3: *$\#(V' \cap \partial[V'])=3$ and V' is neither nearly convex, nor completely stacked.*
Case 4: *V' is nearly convex.*
Case 5: *V' is completely stacked.*

Construction for Case 1. V' fulfills the conditions on V in Theorem 2.2(b), with the vertices v_i' in place of v_i $(1 \leqslant i \leqslant 3)$. As in the proof of 2.2(b) put $V'_{i,i+1} = V' \cap (\mathbb{R}^2 \setminus (v_i' v_{i+1}' \to v_{i+2}'))$ for $1 \leqslant i \leqslant 3$ (all indices are taken modulu 3), and let T' be

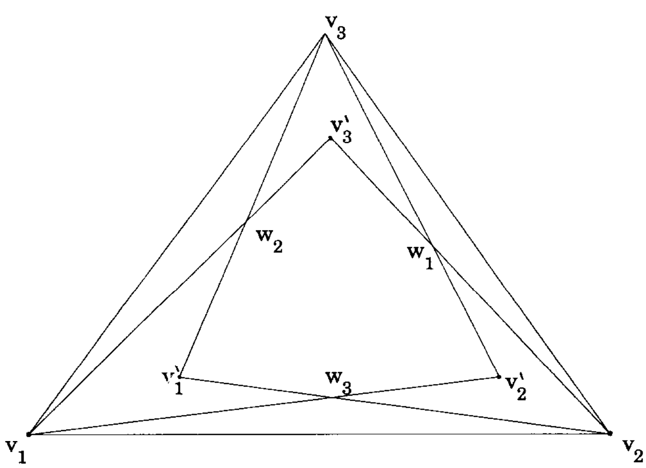

Fig. 4.

a triangulation of $[V']$ whose 0-skeleton is V' s.t. any inner 1-cell of T' with a vertex in $V'_{i,i+1}$ (for some $1 \leqslant i \leqslant 3$) has its other vertex in $v'_i v'_{i+1} \to v'_{i+2}$ (Theorem 2.2(b)). Add all the segments of the form $[v_i, v']$ for $v' \in V'_{i,i+1}$ $(1 \leqslant i \leqslant 3)$ (Fig. 5), to obtain a triangulation T with the desired properties (to the reader).

Construction for Case 2: By Theorem 2.2(a) there is an unstacked convex triangulation T' whose 0-skeleton is V' s.t. any 1-cell of T' whose vertices are on $\partial[V']$ lies entirely on $\partial[V']$ (a boundary cell). Put $V'' = V' \cap \partial[V']$; clearly $v'_i \in V''$ $(1 \leqslant i \leqslant 3)$. As in Case 1 add all segments of the form $[v_i, v'']$ for $v'' \in V'' \cap (\mathbb{R}^2 \setminus (v'_i v'_{i+1} \to v'_{i+2}))$ $(1 \leqslant i \leqslant 3)$ to obtain a triangulation T with the desired properties (to the reader).

Construction for Case 3: Since $\# V' < \# V$ we can use the induction hypothesis to obtain an unstacked convex triangulation T' of $[V']$ whose 0-skeleton is V'.

Claim 1. *Let $\hat{v}_i$ $(1 \leqslant i \leqslant 3)$ be the third vertex of the 2-cell (triangle) of T' which is incident with the edge $[v'_i, v'_{i+1}]$ (lying inside the triangle $[v'_1, v'_2, v'_3]$). Then at least one of the 6 quadruples $\{\hat{v}_i, v'_i, v_i, v'_{i+1}\}$, $\{\hat{v}_i, v'_{i+1}, v_{i+1}, v'_i\}$ $i = 1, 2, 3$ is in a strictly convex position, i.e., its convex hull is a (convex) quadrilateral.*

Proof. If for some $1 \leqslant i \leqslant 3$ $[\hat{v}_i, v'_i, v_i, v'_{i+1}]$ is not a convex quadrilateral and nor is $[\hat{v}_{i-1}, v'_i, v_i, v'_{i-1}]$, then $\hat{v}_i = \hat{v}_{i-1}$ and the points $v_i, v'_i, \hat{v}_i$ $(= \hat{v}_{i-1})$ are collinear. Assume that this happens for $i = 1, 2, 3$ (we have to contradict this assumption). Then $\hat{v}_1 = \hat{v}_2 = \hat{v}_3$; denote this point by $\hat{v}$. Hence $[\hat{v}, v'_i, v'_{i+1}]$ is a 2-cell of T' for $i = 1, 2, 3$, and

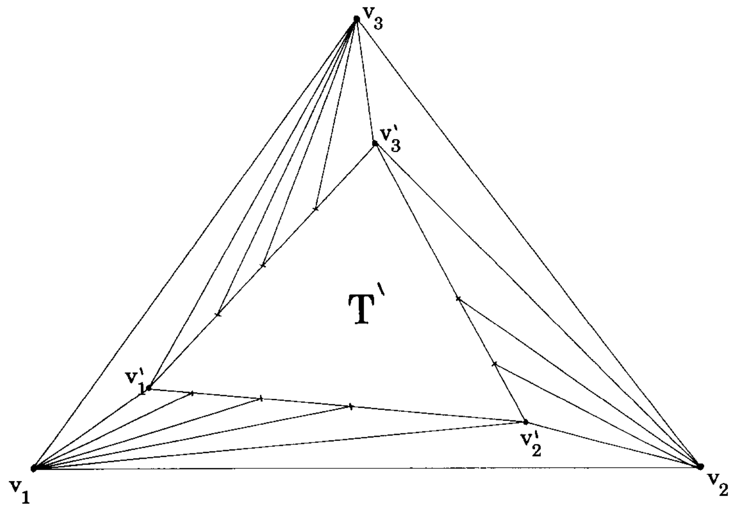

Fig. 5.

it follows that $\# V'=4$, and by the collinearity of $\{v_i, v'_i, \hat{v}\}$ V is completely stacked, contradiction. This proves the claim.

Assume, w.l.o.g., that $\hat{v}_1, v'_1, v_1, v'_2$ are the vertices of a convex quadrilateral (Fig. 6). Construction of T: Leave all 1-cells of T' except for $[v'_1, v'_2]$; replace it by $[v_1, \hat{v}_1]$, and add the segments $[v_i, v]$ for $v \in \{v'_i, v'_{i+1}, v_{i+1}\}$ $1 \leqslant i \leqslant 3$. It is readily checked that T is a triangulation of the desired kind.

Construction for Case 4. V' is nearly convex. Assume, w.l.o.g., that $V' \setminus \{v'_3\}$ is in a weak convex position, and denote by $z_1, \ldots, z_k (k \geqslant 1)$ the vertices of $[V' \setminus \{v'_3\}]$ excluding v'_1 and v'_2, the indexing $1, \ldots, k$ being compatible with the natural order on $\partial[V' \setminus \{v'_3\}]$ from v'_1 to z_1, to z_2, etc., to z_k terminating in v'_2 (see e.g., Fig. 7). Split the discussion into two subcases, $k=1$, and $k>1$.

Subcase 4(i): $k=1$, i.e., $[V' \setminus \{v'_3\}]$ is a triangle and possibly there are more points of V on its edges $[z_1, v'_1]$ and $[z_1, v'_2]$. Since V' is not completely stacked, z_1 misses at least one of the lines $\operatorname{aff}(v_i, v'_i)$ $1 \leqslant i \leqslant 3$. We deal separately with these three cases (the cases $i=1, 2$ being clearly symmetrical it is enough to deal only with one of them, say $i=1$). Start with $i=3$:

Sub-subcase 4(i) 1: $z_1 \notin \operatorname{aff}(v_3, v'_3)$. Then at least one of the quadruples $\{v_3, v'_3, z_1, v'_1\}$ or $\{v_3, v'_3, z_1, v'_2\}$ is in a convex position (strict sense), say, w.l.o.g. (by symmetry), $[v_3, v'_3, z_1, v'_1]$ is a (convex) quadrilateral.

Construction of T (Fig. 7): the inner 1-cells of T are: $\{[v_3, v]\colon v \in \{v'_3\} \cup (V \cap [z_1, v'_1])\}$ union with $\{[v'_3, v]\colon v \in \{v_2\} \cup (V \cap [z_1, v'_2])\}$ union with $\{[v'_2, v]\colon v \in \{v_2, v_1, v'_1\}\}$ union with $\{[v_1, v'_1]\}$ union with the set of 1-cells of any convex triangulation T of the triangle $[V' \setminus \{v'_3\}]$ whose 0-skeleton is $V' \setminus \{v'_3\}$. (T' is *not* specified in Fig. 7) (to the reader).

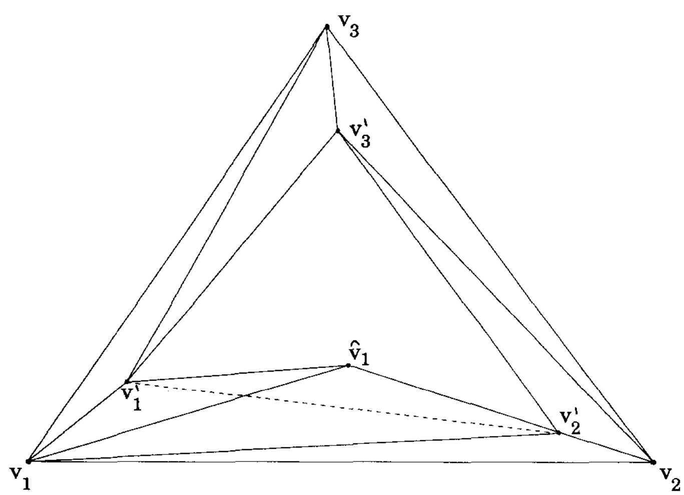

Fig. 6.

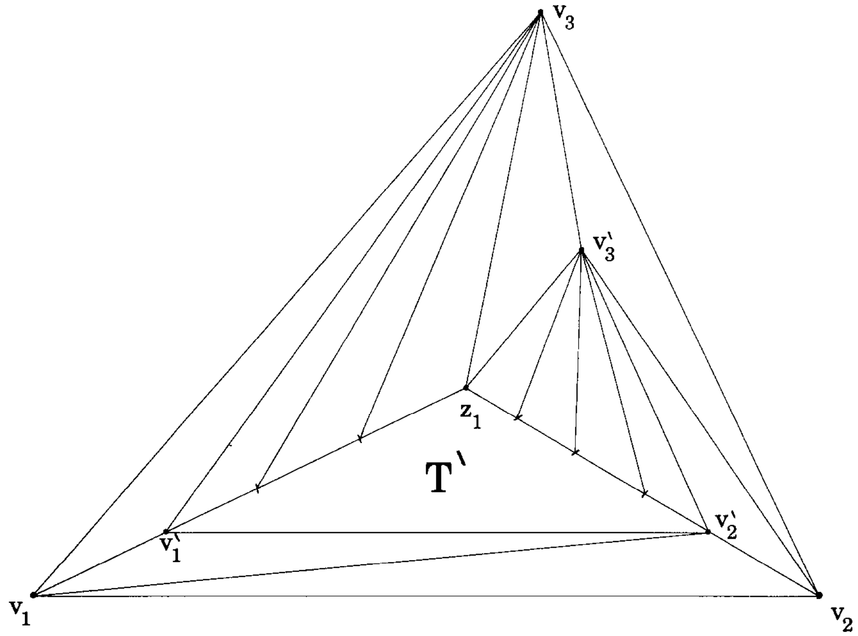

Fig. 7.

Sub-subcase 4(i) 2: $z_1 \in \mathrm{aff}(v_3, v_3')$, and either $z_1 \notin \mathrm{aff}(v_1, v_1')$ or $z_1 \notin \mathrm{aff}(v_2, v_2')$, say (w.l.o.g., because of symmetry), $z_1 \notin \mathrm{aff}(v_1, v_1')$. Then either $\{v_1, v_1', z_1, v_3\}$ or $\{v_1, v_1', z_1, v_2'\}$ is in convex position (strict sense; the 'or' here is exclusive); we deal with these cases separately.

Sub-subcase 4(i) 2(i): ($z_1 \in \mathrm{aff}(v_3, v_3')$ and) $[v_1, v_1', z_1, v_3]$ is a (convex) quadrilateral. Construction of T (Fig. 8). The inner 1-cells of T include: $\{[v_1, v]$: $v \in \{v_3'\} \cup (V \cap [v_1', z_1])\}$ union with $\{[v_3', v]$: $v \in \{v_3\} \cup (V \cap [z_1, v_2'])\}$ union with $\{[v_2', v]$: $v \in \{v_3, v_2, v_1'\}$ union with $\{[v_2, v_1']\}$ union with the set of 1-cells of *any* convex triangulation T' of the triangle $[V' \setminus \{v_3'\}]$ whose vertex set is $V' \setminus \{v_3'\}$ (T' is *not* specified in Fig. 8) (to the reader).

Sub-subcase 4(i) 2(ii): ($z_1 \in \mathrm{aff}(v_3, v_3')$ and) $[v_1, v_1', z_1, v_2']$ is a (convex) quadrilateral. Construction of T (Fig. 9): The inner 1-cells of T are: $\{[v_1, v]$: $v \in (V \cap ([v_1', z_1] \cup [z_1, v_2']))\}$ union with $\{[v_3', v]$: $v \in (V \cap ([v_1', z_1] \cup [z_1, v_2'])) \cup \{v_2, v_3\}\}$ union with $\{[v_3, v_1'], [v_2, v_2']\}$ (to the reader).

Subcase 4(ii): $k > 1$, i.e., $[V' \setminus \{v_3'\}]$ has more than three vertices. Then $z_1 \neq z_k$ and at least one of the quadrangles (v_2', z_k, v_3', v_3) or (v_1', z_1, v_3', v_3) is strictly convex. Put $V'' = V \cap \mathrm{int}[v_1', v_3', z_k]$ and let v'' be the point of $V'' \cap [z_{k-1}, z_k[$ (a half open segment) nearest to z_k. Because of the symmetry between v_1' and v_2' we may assume, w.l.o.g., that $[v_2', z_k, v_3', v_3]$ is a (convex) quadrilateral. The inner 1-cells of T are now $\{[v_3, v]$: $v \in \{v_3'\} \cup (V \cap [z_k, v_2'])\}$, union with $\{[v_2, v]$: $v \in \{v_2', v_1'\}\}$ union with $\{[v_1, v]$: $v \in \{v_1', v_3'\}\}$ union with $\{[v, v_2'], [v, v_3']$: $v \in V''\}$ union with $\{[v'', v]$: $v \in V \cap [z_k, v_2']\}$ (Fig. 10). It is readily checked that T has the desired properties.

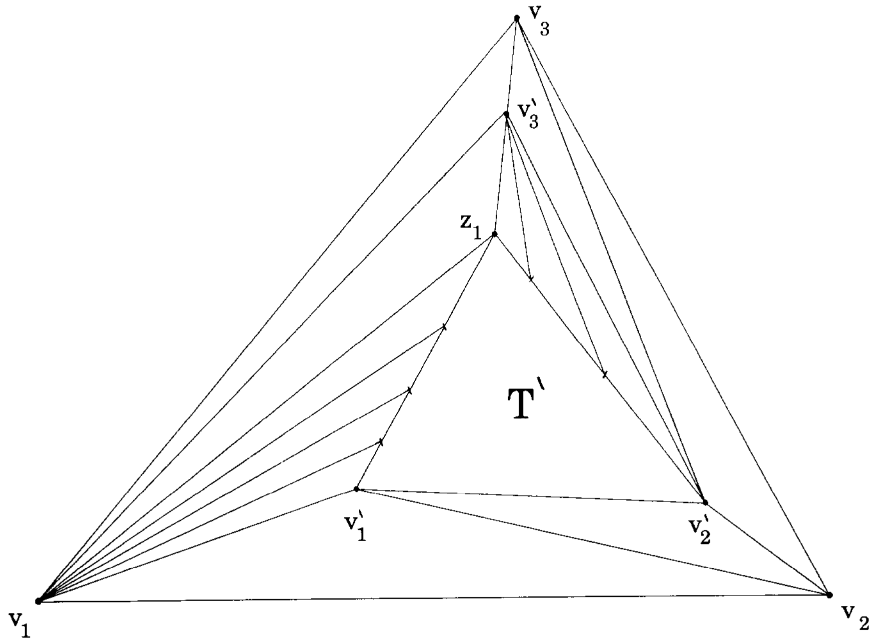

Fig. 8.

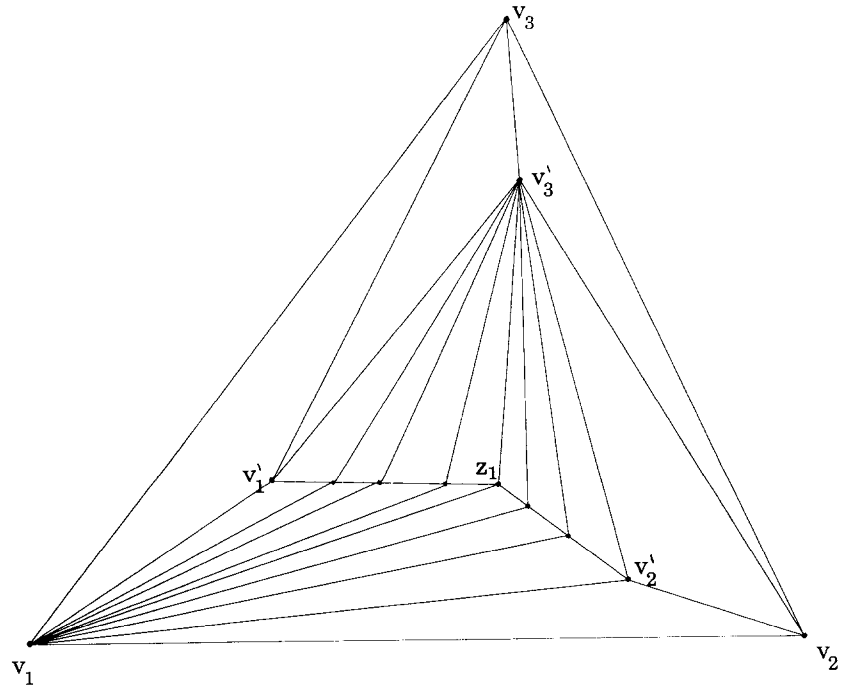

Fig. 9.

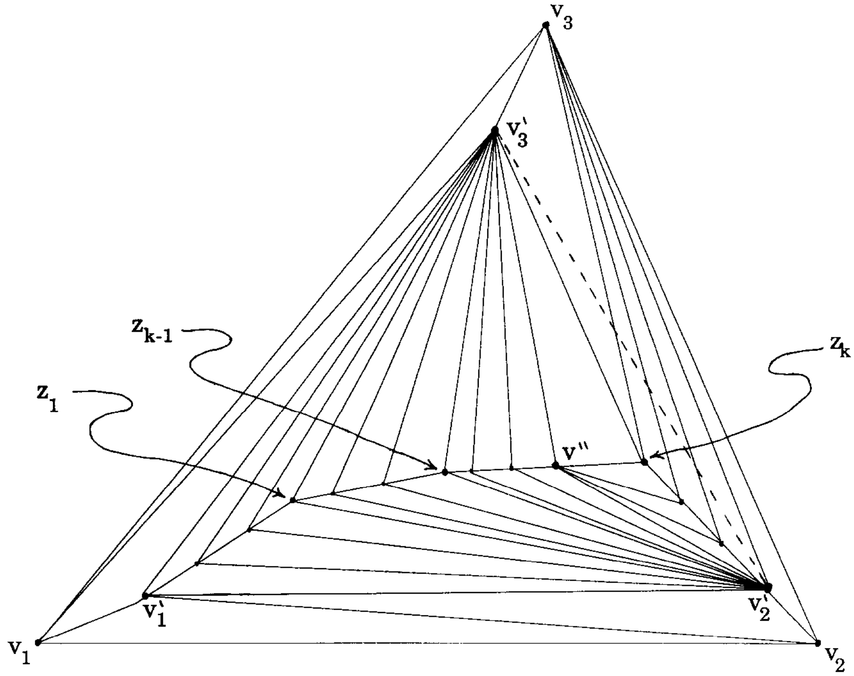

Fig. 10.

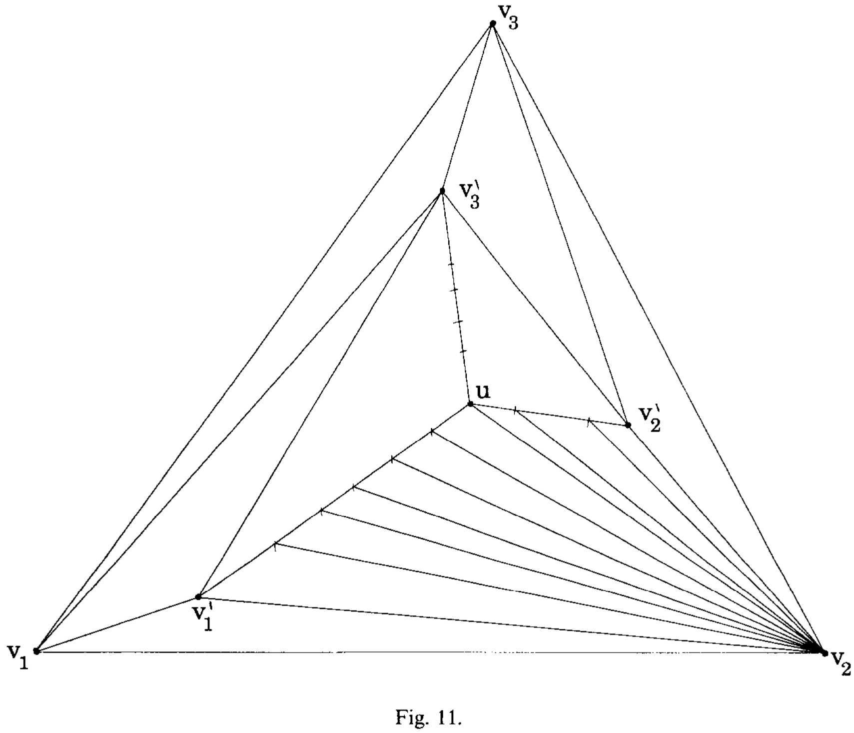

Fig. 11.

Construction for Case 5. V' is completely stacked. Let u be the 'central point' of V', i.e., $u \in V'$ and $V' \subset \bigcup_{i=1}^{3} [u, v_i']$. Since V is not completely stacked at least one of the six quadruples $\{u, v_i', v_i, v_{i+1}'\}$, $\{u, v_i', v_i, v_{i-1}'\}$ $1 \leqslant i \leqslant 3$ is in a convex position. Assume, w.l.o.g., that $[u, v_2', v_2, v_1']$ is a (convex) quadrilateral. The inner 1-cells of T include: $\{[v_2, v]$: $v \in V \cap ([v_2', u] \cup [u, v_1'])\}$ union with $\{[v_1, v]$: $v \in \{v_1', v_3'\}\}$ union with $\{[v_3, v]$: $v \in \{v_3', v_2'\}\}$ union with $\{[v_3', v_2'], [v_3', v_1']\}$ (Fig. 11). Any triangulation of $[V]$ which has V as a 0-dimensional skeleton and which includes the 1-cells described above will satisfy the desired conditions (to the reader). □

Final remark. Which convex triangulations of whose 0-skeleton only three points lie on the boundary are a concrete Schlegel diagram of some simplicial 3-polytope ('concrete' in the sense of the introduction, i.e., via a central projective transformation)? Clearly it suffices to confine the question to unstacked triangulations. There is a simple example of a planar set V of cardinality 6, $\#(V \cap \partial[V]) = 3$, s.t. *any* convex unstacked triangulation on V is *not* a Schlegel diagram of a simplicial 3-polytope. (See [3, Fig. 1, p. 6].) This shows that Theorem 3.1 gives a partial answer only to the question posed above. In 4-space it is known that: there are triangulations of a 3-simplex (tetraeder) which are not the Schlegel diagram of any simplicial 4-polytope, not even in the combinatorial sense (see [4, p. 45, lines 14–18]).

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